

ON THE GENERALIZED CONVEXITY AND CONCAVITY

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ABSTRACT. In this paper, authors study the convexity and concavity properties of real-valued function with respect to the classical means, and prove a conjecture posed by Bruce Ebanks in [12].

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1. INTRODUCTION

A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is $[m_1, m_2]$ -convex (concave) if $f(m_1(x, y)) \leq (\geq) m_2(f(x), f(y))$ for all $x, y \in \mathbb{R}_+ = (0, \infty)$ and $m_1, m_2 \in \mathbb{M}$, where \mathbb{M} denotes the family of all mean values of two numbers in \mathbb{R}_+ . Some examples of mean values of two distinct positive real numbers are given below:

$$\text{Arithmetic mean:} \quad A = A(x, y) = \frac{x + y}{2},$$

$$\text{Geometric mean:} \quad G = G(x, y) = \sqrt{xy},$$

$$\text{Harmonic mean:} \quad H = H(x, y) = \frac{1}{A(1/x, 1/y)},$$

$$\text{Logarithmic mean:} \quad L = L(x, y) = \frac{x - y}{\log(x) - \log(y)},$$

$$\text{Identric mean:} \quad I = I(x, y) = \frac{1}{e} \left(\frac{x^x}{y^y} \right)^{1/(x-y)},$$

$$\text{Alzer mean:} \quad J_p = J_p(x, y) = \frac{p}{p+1} \frac{x^{p+1} - y^{p+1}}{x^p - y^p}, \quad p \neq 0, -1,$$

$$\text{Power mean:} \quad M_t = M_t(x, y) = \begin{cases} \left(\frac{x^t + y^t}{2} \right)^{1/t}, & t \neq 0, \\ \sqrt{x y}, & t = 0. \end{cases}$$

It is easy to see that $J_1(x, y) = A(x, y)$, $J_0(x, y) = L(x, y)$, $J_{-2}(x, y) = H(x, y)$. For the historical background of these means we refer the reader to see [4, 5, 11, 15, 16] and the bibliography of these papers.

Before we introduce the earlier results from the literature we recall the following definition, see [2, 6].

1.1. Definition. Let $f : I \rightarrow (0, \infty)$ be continuous, where I is a sub-interval of $(0, \infty)$. Let M and N be two any mean functions. We say that the function f is MN -convex (concave) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)) \quad \text{for all } x, y \in I.$$

In [2], Anderson, Vamanamurthy and Vuorinen studied the convexity and concavity of a function f with respect two mean values, and gave the following detailed result:

1.2. Theorem. Let I be an open sub-interval of $(0, \infty)$ and let $f : I \rightarrow (0, \infty)$ be differentiable. In parts (4)(9), let $I = (0, b)$, $0 < b < \infty$.

- (1) f is AA -convex (concave) if and only if $f'(x)$ is increasing (decreasing),
- (2) f is AG -convex (concave) if and only if $f'(x)/f(x)$ is increasing (decreasing),
- (3) f is AH -convex (concave) if and only if $f'(x)/f(x)^2$ is increasing (decreasing),
- (4) f is GA -convex (concave) if and only if $xf'(x)$ is increasing (decreasing),
- (5) f is GG -convex (concave) if and only if $xf'(x)/f(x)$ is increasing (decreasing),
- (6) f is GH -convex (concave) if and only if $xf'(x)/f(x)^2$ is increasing (decreasing),
- (7) f is HA -convex (concave) if and only if $x^2f'(x)$ is increasing (decreasing),
- (8) f is HG -convex (concave) if and only if $x^2f'(x)/f(x)$ is increasing (decreasing),
- (9) f is HH -convex (concave) if and only if $x^2f'(x)/f(x)^2$ is increasing (decreasing).

After the publication of [2] many authors have studied generalized convexity. For a partial survey of the recent results, see [3].

In [9], the following inequalities were studied:

1.3. Theorem. Let $f : I \rightarrow (0, \infty)$ be a continuous and $I \subseteq (0, \infty)$, then

- (1) f is LL -convex (concave) if f is increasing and \log -convex (concave),
- (2) f is AL -convex (concave) if f is increasing and \log -convex (concave),

Recently, Baricz [7] took one step further and studied the MN -convexity (concavity) of a function f in a generalized way, and gave the following result:

1.4. Lemma. [7, Lemma 3] Let $p, q \in \mathbb{R}$ and let $f : [a, b] \rightarrow (0, \infty)$ be a differentiable function for $a, b \in (0, \infty)$. The function f is (p, q) -convex ((p, q) -concave) if and only if $x \mapsto x^{1-p}f'(x)(f(x))^{q-1}$ is increasing (decreasing).

It can be observed easily that $(1, 1)$ -convexity means the AA -convexity, $(1, 0)$ -convexity means the AG -convexity, and $(0, 0)$ -convexity means GG -convexity.

1.5. Lemma. [7, Theorem 7] *Let $a, b \in (0, \infty)$ and $f: [a, b] \rightarrow (0, \infty)$ be a differentiable function. Denote $g(x) = \int_1^x f(t) dt$ and $h(x) = \int_x^b f(t) dt$. Then*

(a) If for all $p \in [0, 1]$ the function f is $(p, 0)$ -concave, then the function g is (p, q) -concave for all $p \in [0, 1]$ and $q \leq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is increasing for all $p \in [0, 1]$, then g is (p, q) -concave for all $p \in [0, 1]$ and $q \in (0, 1)$. Moreover, if for all $p \in \mathbb{R}$ the function $x \mapsto x^{1-p}f(x)$ is increasing, then g is (p, q) -convex for all $p \in \mathbb{R}$ and $q \geq 1$.

(b) If for all $p \in [0, 1]$ the function f is $(p, 0)$ -concave, then the function g is (p, q) -concave for all $p \in [0, 1]$ and $q \leq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is decreasing for all $p \in [0, 1]$, then g is (p, q) -concave for all $p \in [0, 1]$ and $q \in (0, 1)$. Moreover, if for all $p \in \mathbb{R}$ the function $x \mapsto x^{1-p}f(x)$ is decreasing, then g is (p, q) -convex for all $p \in \mathbb{R}$ and $q \geq 1$.

(c) If for all $p \notin (0, 1)$ we have $a^{1-p}f(a) = 0$ and the function f is $(p, 0)$ -convex, then g is (p, q) -convex for all $p \notin (0, 1)$ and $q \geq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is increasing for all $p \notin (0, 1)$, then g is (p, q) -convex for all $p \notin (0, 1)$ and $q < 0$.

(d) If for all $p \notin (0, 1)$ we have $b^{1-p}f(b) = 0$ and the function f is $(p, 0)$ -convex, then g is (p, q) -convex for all $p \notin (0, 1)$ and $q \geq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is decreasing for all $p \notin (0, 1)$, then g is (p, q) -convex for all $p \notin (0, 1)$ and $q < 0$.

In this paper we make a contribution to the subject by giving the following theorems.

1.6. Theorem. *Let $f : I \rightarrow (0, \infty)$ and $I \subseteq (0, \infty)$. Then the following inequality holds true:*

$$\begin{aligned} I(f(x), f(y)) &\geq f(I(x, y)) \\ I(f(x), f(y)) &\leq f(A(x, y)), \end{aligned}$$

if the function $f(x)$ is a continuously differentiable, increasing and log-convex(concave).

1.7. Theorem. *Let f be a continuous real-valued function on $(0, \infty)$. If f is strictly increasing and convex, then*

$$(1.8) \quad P_f(x, y) \leq R_f(x, y)$$

where

$$P_f(x, y) = f \left((xy)^{1/4} \left(\frac{x+y}{2} \right)^{1/2} \right)$$

and

$$R_f(x, y) = \frac{1}{y-x} \int_x^y f(t) dt.$$

1.9. Remark. In [12], Ebanks defined $P_f(x, y)$ and $R_f(x, y)$, and proposed an open problem for a continuous and strictly monotonic real-valued function f on $(0, \infty)$ as follows:

Problem. Does f strictly increasing and convex (or $f'' > 0$) imply $P_f \leq R_f$?

It is obvious that the Theorem 1.7 gives an affirmative answer to the Ebanks' problem.

1.10. Theorem. *Let $f : I \rightarrow (0, \infty)$ and $I \subseteq (0, \infty)$.*

(1) If $f(x)$ is continuously differentiable, strictly increasing(decreasing) and convex(concave) and $f^{p-1}(x)f(x)$ is increasing on $(0, 1)$, then

$$J_p(f(x), f(y)) \geq f(J_p(x, y))$$

$$J_p(f(x), f(y)) \leq f(A(x, y))$$

for $p \leq 1$.

(2) If $f(x)$ is continuously differentiable, strictly decreasing(increasing) and convex(concave) and $f^{p-1}(x)f(x)$ is decreasing on $(0, 1)$, then

$$J_p(f(x), f(y)) \geq f(J_p(x, y))$$

$$J_p(f(x), f(y)) \leq f(A(x, y))$$

for $p > 1$.

2. LEMMAS AND PROOFS

We recall the following lemmas which will be used in the proof of the theorems.

2.1. Lemma. [17] *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \rightarrow \mathbb{R}$ be a positive, integrable function. Then*

$$(2.2) \quad \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx.$$

If one of the functions f or g is non-increasing and the other non-decreasing, then the inequality in (3.1) is reversed.

2.3. Lemma. [13] *If $f(x)$ is continuous and convex function on $[a, b]$, and $\varphi(x)$ is continuous on $[a, b]$, then*

$$(2.4) \quad f\left(\frac{1}{b-a} \int_a^b \varphi(x)dx\right) \leq \frac{1}{b-a} \int_a^b f(\varphi(x))dx.$$

If function $f(x)$ is continuous and concave on $[a, b]$, the inequality in (2.4) is reversed.

2.5. Lemma. [5] *Fix two positive number a, b . Then $L(a, b) \leq I(a, b) \leq A(a, b)$.*

2.6. Lemma. [13] *The function $p \mapsto J_p(x, y)$ is strictly increasing on $\mathbb{R} \setminus \{0, -1\}$.*

Proof of Theorem 1.6. Since the proof of part (2) is similar to part (1), we only prove the part (1) here. An easy computation and substitution $t = f(u)$ yield

$$(2.7) \quad \ln I(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} \ln t dt}{\int_{f(y)}^{f(x)} 1} = \frac{\int_y^x \ln f(u) f'(u) du}{\int_y^x f'(u) du}.$$

Since the functions $f(x)$ and $f'(x)$ are increasing on $I \subseteq (0, \infty)$, now by using Lemma 2.1 we have

$$(2.8) \quad \int_y^x 1 du \cdot \int_y^x \ln f(u) f'(u) du \geq \int_y^x f'(u) du \cdot \int_y^x \ln f(u) du.$$

Combining (2.7) and (2.8), we obtain

$$I(f(x), f(y)) \geq \frac{\int_y^x \ln f(u) du}{y - x}.$$

Considering the log-convexity of the function $f(x)$ and using Lemmas 2.3 and 2.5, we get

$$I(f(x), f(y)) \geq \ln f \left(\frac{\int_y^x u du}{y - x} \right) = \ln f \left(\frac{x + y}{2} \right) \geq \ln f(I(x, y)).$$

This completes the proof. \square

Proof of Theorem 1.7. Since f is strictly increasing and convex, by utilizing the Lemma 2.1 and the inequality $G(x, y) \leq A(x, y)$ we obtain

$$\begin{aligned} R_f(x, y) &\geq \frac{\int_x^y f(u) du}{y - x} \geq f \left(\frac{\int_x^y u du}{y - x} \right) \\ &= f \left(\frac{x + y}{2} \right) \geq f \left((xy)^{1/4} \left(\frac{x + y}{2} \right)^{1/2} \right) \\ &= P_f(x, y). \end{aligned}$$

This completes the proof. \square

Proof of Theorem 1.10. For the proof of part (1), letting $t = f(u)$, we get

$$J_p(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} t^p dt}{\int_{f(y)}^{f(x)} t^{p-1}} = \frac{\int_y^x f^p(u) f'(u) du}{\int_y^x f^{p-1}(u) f'(u) du}.$$

By using Lemma 2.1, we obtain

$$J_p(f(x), f(y)) \geq \frac{\int_y^x f(u) du}{y - x}.$$

Considering convexity of the function $f(x)$ and using Lemmas 2.3 and 2.6, we get

$$J_p(f(x), f(y)) \geq f \left(\frac{\int_y^x u du}{y - x} \right) = f \left(\frac{x + y}{2} \right) \geq f(J_p(x, y)),$$

this implies (1). The proof of part (2) follows similarly. \square

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